UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH3506**
ASSESSMENT PATTERN	:	**MATH3506A**
MODULE NAME	:	**Mathematical Ecology**
DATE	:	**08-May-14**
TIME	:	**14:30**
TIME ALLOWED	:	**2 Hours 0 Minutes**

2013/14-MATH3506A-001-EXAM-24

TURN OVER

All questions may be attempted but only marks obtained on the best four *solutions will count.*
The use of an electronic calculator is not *permitted in this examination.*

1. Two interacting species with densities x and y are modelled by the system

$$\begin{aligned} \frac{dx}{dt} &= x(a - bx - cy), \\ \frac{dy}{dt} &= y(-d + ex - fy), \end{aligned} \tag{1}$$

where $a, b, c, d, e, f > 0$.

(a) Briefly discuss the model, identifying the type of species-species interactions involved.

(b) Find all steady states of the system (1) and determine whether they are locally stable or unstable.

(c) Sketch the phase planes for the system (1) when $ae < bd$.

(d) Describe the possible time evolutions of the predator density when $b = 0 = f$.

2. A predator-prey model has the form

$$\begin{aligned} \frac{dN_1}{dt} &= \rho N_1 \left(1 - \frac{N_1}{K}\right) - \phi(N_1, N_2), \\ \frac{dN_2}{dt} &= N_2(\sigma N_1 - \mu), \end{aligned} \tag{2}$$

where $\phi(N_1, N_2) = \dfrac{\gamma N_1 N_2}{A + N_1}$ and $\rho, K, \gamma, A, \sigma$ are all positive constants.

(a) Which of N_1 and N_2 represents the predator, and which the prey?

(b) Sketch $\phi(N_1, N_2)$ for a fixed $N_2 > 0$. What does the parameter γ represent?

(c) Find all steady states of (2) and determine whether they are locally stable or unstable.

(d) Show that when $K\sigma > 2\mu$, a limit cycle is possible around the interior steady state as A varies and find the critical value A_c of A at which it occurs.

(e) Sketch the phase plane for (2) when A is just less than A_c.

3. A fish population is modelled by the discrete time system

$$x_{t+1} = f(x_t), \quad t = 0, 1, 2, \dots, \tag{3}$$

where f is a smooth real-valued function defined on $[0, \infty)$.

(a) Write down the equation that x^* must satisfy if it is a steady state population of (3). What is the condition that a steady state x^* is locally asymptotically stable?

(b) When $f(x) = \dfrac{rx}{1+x^3}$, where $r > 0$, sketch the cobweb map for (3) for the cases $r < 1$ and $r > 1$.

(c) The fish population is now harvested, so that $f(x) = \dfrac{rx}{1+x^3} - x$. Sketch the cobweb maps for the cases $r < 3$ and $r > 3$, and comment briefly on your results.

4. In an age-structured population there are n age classes and the density at age k at time t is $N_k(t)$. Here $n > 0$ is the maximum age of any individual. The expected number of offspring to females of age k is b_k, for $k = 1, \dots, n$, and the probability that an individual of age k, where $0 \leqslant k \leqslant n-1$ (with $k = 0$ newborns), survives to age $k+1$ is p_k.

(a) Show that $N(t+1) = LN(t)$, where $N(t) = (N_1(t), \dots, N_n(t))$ and L is some $n \times n$ real matrix which you should find.

(b) Derive the Euler–Lotka equation for the eigenvalues of L.

(c) In the case that $b_k = 0$, for $k = 1, \dots, n-1$, and $b_n = b > 0$, show that

$$N(t+n) = \omega N(t), \quad t = 0, 1, \dots,$$

where $\omega > 0$ is a constant which you should find. Hence determine how the age distribution of the population changes qualitatively with time.

(d) Find the critical birth rate below which the population eventually dies out.

5. (a) Find the explicit solution of the logistic equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \quad N(0) = N_0,$$

where $r, K > 0$ are constants.

(b) Consider the time-dependent logistic equation

$$\frac{dN}{dt} = \rho(t)N\left(1 - \frac{N}{K}\right), \quad N(0) = N_0,$$

with a function ρ periodic with period T. By introducing the new rescaled time variable τ via $d\tau = \rho(t)dt$, or otherwise, show that for $t = nT + s$, with $s \in [0, T)$ and an integer n, the solution of the above equation is given by

$$N(nT + s) = \frac{N_0}{\frac{N_0}{K} + e^{-nRT}\left(1 - \frac{N_0}{K}\right)\exp\left(-\int_0^s \rho(t)\, dt\right)},$$

where $R = \frac{1}{T}\int_0^T \rho(t)dt$.

(i) What is the long term population density in the cases

A. $R > 0$,

B. $R < 0$?

(ii) When $R = 0$, describe qualitatively the possible solutions $N(t)$ for

A. $N(0) < K$,

B. $N(0) > K$.